

A SEMPLE APPROACH TO GOURSAT: THE MULTI-FLAG CASE. UNE APPROCHE SEMPLE À GOURSAT: LE CAS MULTI-DRAPEAU.

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ABSTRACT. We consider the problem of classifying the orbits within a tower of fibrations with fibers diffeomorphic to projective planes and generalize the tower of fiber bundles due to J. Semple. This tower was rediscovered by Montgomery and Zhitomirskii in the context of subriemannian geometry. This tower admits a natural action of the diffeomorphism group of affine 3-space and these orbits correspond to classes of Goursat multi-flags. We demonstrate that it is possible to classify many of these orbits by elementary means by appealing to some basic tools in projective geometry, and the combinatorics of spatial curves.

RÉSUMÉ. Nous nous intéressons au problème de classification des orbites dans une tour de fibration, où les fibres sont difféomorphes à des plans projectifs, généralisant ainsi les tours de fibrés projectives de J. Semple. Cette tour a été redécouverte par R. Montgomery et M. Zhitomirskii dans le contexte de la géométrie sub-riemannienne. Cette tour admet une action naturelle du groupe des difféomorphismes de l'espace affine de dimension 3 et ces orbites correspondent à des classes de multi-drapeaux de Goursat. Nous démontrons qu'il est possible de classer un grand nombre de ces orbites de manière élémentaire en utilisant des outils classiques de géométrie projective et la combinatoire des courbes spatiales.

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1. INTRODUCTION

In this note we generalize the Semple tower from enumerative algebraic geometry ([15], [10], [7]) to study the local theory of Goursat multi-flags, a class of nonholonomic distributions that generalizes the contact distributions of $J(n; k)$, where $J(n; k)$ is the space of k -jets of maps from \mathbb{R} to \mathbb{R}^n .

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A Goursat multi-flag is a nonholonomic distribution D with *slow growth*. More concretely, a Goursat n -flag of length k is a distribution D of rank $(n+1)$ sitting in a $(n+1) + kn$ dimensional ambient manifold where the rank of the associated flag of distributions

$$D \subset D + [D, D] \subset D + [D, D] + [[D, D], [D, D]] \subset \dots$$

grows by n at each bracketing step. One often considers a local family of vector fields that locally spans the distribution D . For the case of $n = 1$ and $n = 2$ there is a concrete Pfaffian description of Goursat multi-flags as the vanishing of locally defined 1-forms. Cartan proved that a generic germ of a Goursat 1-flag can always be described by the vanishing of the 1-forms

$$\omega_1 = dy - u_1 dx, \omega_2 = du_1 - u_2 dx, \dots, \omega_n = du_{n-1} - u_n dx$$

where (x, y, u_1, \dots, u_n) are local coordinates on $R^2 \times (S^1)^n$ and showed that any two generic Goursat germs are equivalent to the germ of the distribution described by these 1-forms ([4]). However, A. Kumpera and C. Ruiz pointed out in [9] that there are Goursat 1-flags that are not locally equivalent to the vanishing of the 1-forms listed above and are referred to as *singularities*. One way to determine these 1-forms is by using a coding system known as *RVT* code developed in [11] and [6].

In [5], we worked with a tower of fiber bundles, denoted by $S(n; k)$, containing the space $J(n; k)$ as an open dense subset. Each space $S(n; k)$ comes equipped with a geometric distribution Δ_k . The link between Goursat multi-flags and the Semple tower is the following :

the problem of classifying Goursat multi-flags up to local equivalence is equivalent to the classification of points in the Semple tower $S(n; k)$ up to symmetry.

Our classification is only for Goursat multi-flags of short length. In this note we specialize to the study of $\text{Diff}(\mathbb{A}^3)$ -orbits in $S(2; 4)$. This is equivalent to studying local normal forms of Goursat 2-flags of length up to 4. Our proofs are constructive and rely upon elementary projective geometry and combinatorial invariants of spatial curves ([1]). For the determination of local orbits we introduce the so called *isotropy method*.

2. DEFINITIONS AND STATEMENT OF MAIN RESULTS

A *geometric distribution* hereafter denotes a linear subbundle of the tangent bundle.

2.1. Prolongation. Let the pair (Z, Δ) denote a manifold Z of dimension d equipped with a distribution Δ of rank r . We denote by $\mathbb{P}(\Delta)$ the *projectivization* of Δ . As a manifold,

$$\mathbb{P}(\Delta) := Z^1,$$

has dimension $d + (r - 1)$.

Given an analytic horizontal curve $c : (I, 0) \rightarrow (Z, q)$, where I is some open interval in \mathbb{R} containing the origin and $c(0) = q$, we can naturally define a new curve

$$c^1 : (I, 0) \rightarrow (Z^1, (q, \ell))$$

with image in Z^1 and where $\ell = \text{span}\{\frac{dc}{dt}(0)\} \subset \Delta_q$. This new curve, $c^1(t)$, is called the *prolongation* of $c(t)$. This procedure can be iterated.

TABLE 1. Some geometric objects and their Cartan prolongations.

TABLEAU 1. Certains objets géométriques et leurs prolongements de Cartan.

curve $c : (I, 0) \rightarrow (Z, q)$	curve $c^1 : (I, 0) \rightarrow (Z^1, q)$, $c^1(t) = (\text{point, moving line}) = (c(t), \text{span}\{\frac{dc}{dt}(t)\})$
diffeomorphism $\Phi : Z \curvearrowright$	diffeomorphism $\Phi^1 : Z^1 \curvearrowright$, $\Phi^1(p, \ell) = (\Phi(p), d\Phi_p(\ell))$
rank r linear subbundle $\Delta \subset TZ$	rank r linear subbundle $\Delta_{1(p, \ell)} = d\pi_{(p, \ell)}^{-1}(\ell) \subset TZ^1$, $\pi : Z^1 \rightarrow Z$ is the canonical projection.

The manifold Z^1 also comes equipped with a distribution Δ_1 called the *Cartan prolongation of Δ* ([2]) which is defined as follows. Let $\pi : Z^1 \rightarrow Z$ be the projection map $(p, \ell) \mapsto p$. Then

$$\Delta_1(p, \ell) = d\pi_{(p, \ell)}^{-1}(\ell),$$

It is easy to check that Δ_1 is also a distribution of rank r .

By a *symmetry* of the pair (Z, Δ) we mean a local diffeomorphism Φ of Z that preserves the subbundle Δ .

The symmetries of (Z, Δ) can also be prolonged to symmetries Φ^1 of (Z^1, Δ_1) as follows. Define

$$\Phi^1(p, \ell) := (\Phi(p), d\Phi_p(\ell)).$$

Since $d\Phi_p$ is invertible, the second component is well defined as a projective map. This new transformation in (Z^1, Δ_1) is the *prolongation* of Φ . Objects of interest and their Cartan prolongations are summarized in Table 1.

2.2. Constructing the Semple Tower.

Definition 2.1. The *Semple Tower* is a sequence of manifolds with distributions, $(S(n; k), \Delta_k)$, together with fibrations

$$\cdots \rightarrow S(n; k) \rightarrow S(n; k-1) \rightarrow \cdots \rightarrow S(n; 1) \rightarrow S(n; 0) = \mathbb{A}^{n+1}$$

and we write $\pi_{k,i} : S(n; k) \rightarrow S(n; k)$ for the respective bundle projections.

Definition 2.2. $\text{Diff}(n)$ is taken to be the *pseudogroup of diffeomorphism germs of \mathbb{A}^n* .

Theorem 2.1. For $n > 1$ and $k > 0$ any local diffeomorphism of $S(n; k)$ preserving the distribution Δ_k is the restriction of the k -th prolongation of a local diffeomorphism $\Phi \in \text{Diff}(n)$.

Proof : See [16], pg. 795. \square

Shibuya and Yamaguchi also point out that this is a result due to A. Bäcklund ([3]).

Remark 2.2. From now on we will write $S(k)$ to denote $S(2; k)$.

1. We also use the notation Φ_* for the pushforward or tangent map $d\Phi$.

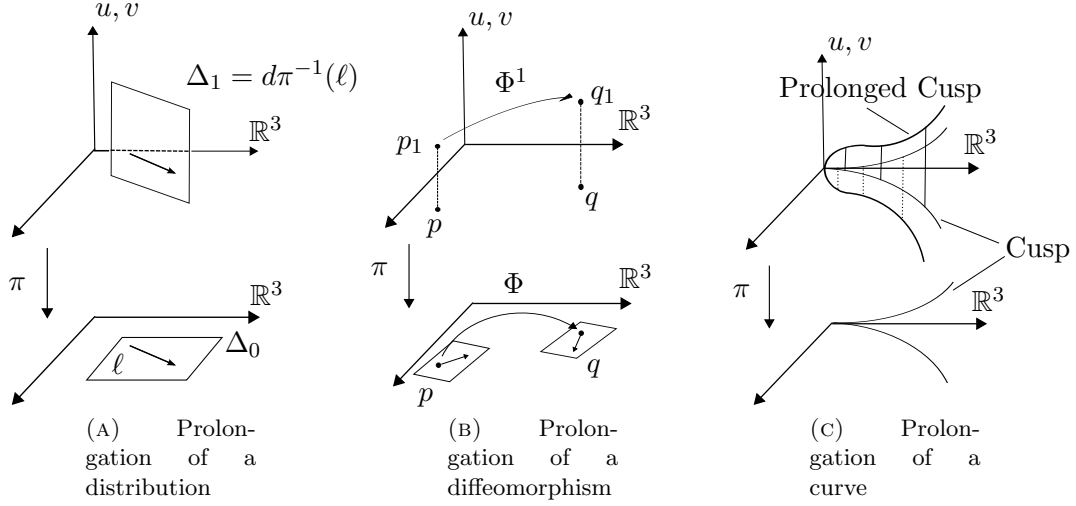


FIGURE 1. Prolongations.
FIGURE 1. Prolongements.

Definition 2.3. Two points p, q in $S(k)$ are said to be *equivalent*, written $p \sim q$, if there is a $\Phi \in \text{Diff}(3)$ such that $\Phi^k(p) = q$.

Definition 2.4. Let $p \in S(k)$ then we denote by $\mathcal{O}(p)$ to be the *orbit of the point p* under the action by elements of $\text{Diff}(3)$ on the k -th level of the Semple Tower. This means that a point q is an element in $\mathcal{O}(p)$ if q is equivalent to the point p .

Definition 2.5. We say that a curve or curve germ $\gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{A}^3, p_0)$ *realizes* the point $p_k \in S(k)$ if $\gamma^k(0) = p_k$, where $p_0 = \pi_{k,0}(p_k)$.

3. MAIN RESULTS AND IDEAS OF PROOFS

From our recent work in [5] we have been able to completely classify the points within the first four levels of the tower. Our findings are summarized by the following.

Theorem 3.1 (Orbit counting per level). In the $n = 2$ (or spatial) Semple Tower the number of orbits within each of the first four levels of the tower are as follows :

- Level 1 has 1 orbit,
- Level 2 has 2 orbits,
- Level 3 has 7 orbits,
- Level 4 has 34 orbits.

In order to classify each of these orbits we used a coding system known as *RVT* coding which partitions the points into equivalence classes that are invariant under the action of the symmetries at any level. The details of this coding system can be found in [6] and [5].

TABLE 2. Number of orbits within the first three levels of the Semple Tower.

TABLEAU 2. Nombre d'orbites dans les trois premiers niveaux de la tour de Semple.

Level of tower	<i>RVT</i> code	Number of orbits	Normal forms
1	<i>R</i>	1	$(t, 0, 0)$
2	<i>RR</i>	1	$(t, 0, 0)$
	<i>RV</i>	1	$(t^2, t^3, 0)$
3	<i>RRR</i>	1	$(t, 0, 0)$
	<i>RRV</i>	1	$(t^2, t^5, 0)$
	<i>RVR</i>	1	$(t^2, t^3, 0)$
	<i>RVV</i>	1	$(t^3, t^5, t^7), (t^3, t^5, 0)$
	<i>RVT</i>	2	$(t^3, t^4, t^5), (t^3, t^4, 0)$
	<i>RVL</i>	1	(t^4, t^6, t^7)

Theorem 3.2 (Listing of orbits within each *RVT* code). Table 2 is a breakdown of the number of orbits that appear within each *RVT* class within the first three levels. For level 4 there is a total of 23 possible *RVT* classes. Of the 23 possibilities 14 of them consist of a single orbit. The classes *RRVT*, *RVRV*, *RVVR*, *RVVV*, *RVVT*, *RVTR*, *RVTV*, and *RVTL* consist of 2 orbits, and the class *RVTT* consists of 4 orbits.

Remark 3.3. The curve normal forms were computed by turning \mathbb{A}^3 into the vector space \mathbb{R}^3 and doing our computations over the origin in \mathbb{R}^3 . More details about the computations and properties of the curve normal forms can be found in [5].

One recent development that was not included in [5] is the following.

Theorem 3.4. Let ω be an *RVT* class that has a total of k orbits. Then the addition of R 's to the beginning of the code ω , meaning $R \cdots R\omega$, will be an *RVT* class with k -orbits.

Theorem 3.4 allows one to reduce the number of calculations needed to compute the number of orbits within an *RVT* class found within higher levels of the Semple Tower.

3.1. Idea of Proofs. When we began our classification of the points within the Semple Tower we followed the approach taken by Montgomery and Zhitomirskii in [11] by using the singular curves approach as well as the *RVT* coding system, which partitioned the various points within the tower ([11], [6]). The curve approach yielded the complete classification of the points of the plane Semple Tower $S(1; k)$. However, while classifying the points within level 4 of the tower we noticed that our curve approach broke down very quickly. This lead us to develop the *isotropy method*. While this technique does not yield the same normal forms as the curve approach, it did help us gain a better overall picture of some of the subtleties about the Semple Tower. In particular, this method helped us understand the incidence relations between the various critical hyperplanes and how these critical directions arise within the tower ([6], [5]). One other important feature of the isotropy method is that it can be used to classify the points within any *RVT* class within the Semple Tower.

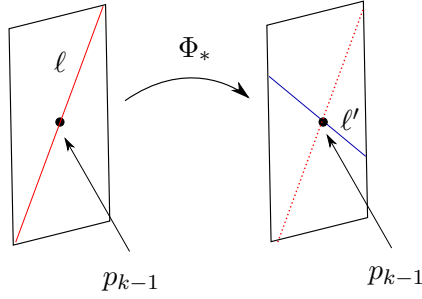


FIGURE 2. The isotropy method.

FIGURE 2. Utilisation de l'isotropie de l'action de groupe pour compter orbites.

3.2. The isotropy method.

Suppose we want to look at a particular *RVT* class, at the k -th level, given by ω (a word of length k) and we want to see how many orbits there are. Suppose as well that we understand its projection $\pi_{k,k-1}(\omega)$ one level down, which decomposes into N orbits. Choose representative points $p_i, i = 1, \dots, N$ for the N orbits in $\pi_{k,k-1}(\omega)$, and consider the group $G_{k-1}(p_i)$ of level $k-1$ symmetries that fix p_i . This group is called the *isotropy group of p_i* . Since elements Φ^{k-1} of the isotropy group fix p_i , their prolongations $\Phi^k = (\Phi^{k-1}, \Phi_*^{k-1})$ act on the fiber over p_i . Under the action of the isotropy group the fiber decomposes into some number $n_i \geq 1$ (possibly infinite) of orbits. Summing up, we find that ω decomposes into $\sum_{i=1}^N n_i \geq N$ orbits. This will tell us how many orbits there are for the class ω .

This is the idea behind the approach (see Figure 2). We want to point out that Theorem 2.1 is a crucial result which allows our isotropy method to work.

Remark 3.5 (The class *RVVV*). In our work we carried out the explicit details of how one uses the isotropy method to calculate the number of orbits. In particular, we compute the orbits of the *RVT* class *RVVV* in explicit detail using the isotropy method. One finds there is a total of 2 orbits within the class *RVVV*.

4. CONCLUSION

Our work with the Semple Tower has been primarily focussed on classifying points up to symmetry at various levels within the tower. As was pointed out in [11], it is possible to end up with moduli appearing within various *RVT* classes, meaning that there is a whole continuum's worth of points in a particular class that are inequivalent. We are interested in determining at which level does this first occur in the $n = 2$ Semple Tower. We believe that one starting point is to look at the *RVT* classes of the form $RV^k R^l V R^m$ for k, l, m positive integers.

In a series of papers by J.C. Hausmann and E. Rodriguez ([14], [8]) they examine a control theory problem known as the *Snake Charmer Algorithm*. This is the continuous analogue of the polygonal snakes that were initially studied by Hausmann in [8]. Due to

the fact that the configuration space for snakes involves looking at continuous C^1 -curves on S^n , Hausmann and Rodriguez take the group action on the configuration space to be the Möbius group. This problem also shares a connection to the world of Goursat-multi flags. Pelletier and Slayman in [13] show that Hausmann's polygonal snakes correspond to the k -bar system and establishes a connection between the distribution arising from the configuration space of snakes to the theory of Goursat multi-flags ([12]). We are interested in extending their work using geometric mechanics to understand what sort of geometric phenomena can appear when trying to control a polygonal snake.

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